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TRANSITIVE ANOSOV FLOWS AND PSEUDO-ANOSOV MAPS

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A TRANSITIVE Anosov flow on a closed manifold M is one with the qualitative behavior of a geodesic flow on a surface of negative curvature, that is global hyperbolicity and dense periodic set. A pseudo-Anosov map is a homeomorphism of a closed surface that has finitely many prescribed prong singularities and is smooth and hyperbolic elsewhere: we refer to the Orsay Thurston Seminar for details[2]. We will show that Birkhoff's surfaces of section[1] can be used to establish a close connection between these systems when M has dimension 3. This extends the surgery techniques of [4, 5] to produce all the transitive Anosov flows in dimension 3.

§1. THE CONSTRUCTION OF ANOSOV FLOWS

Let S be a closed surface and $f: S \circlearrowright$ a pseudo-Anosov map. At each singularity p of f we have $2k$ prongs, $k = k(p) \geq 3$. Let $m = m(p)$ be the least period of p. We assume f^m preserves the local orientation at p. There is a smallest positive integer $j = j(p)$ such that f^{mj} carries each prong at p into itself. We have $j|k$ and we make the assumption that for all singularities p, $k/j \in \{1, 2\}$. Let F be a finite f-invariant set that contains all the singularities of f.

We construct a flow φ^* as follows. Let S^* be the surface obtained from S by blowing up each point $p \in F$ using polar coordinates. Then S^* has one boundary component for each p. There is a map $f^*: S^* \circlearrowright$ determined by f that is semiconjugate to f under the collapsing map $S^* \to S$. This diffeomorphism f^* is a pseudo-Anosov map with all its singularities on ∂S^*. We let M^* be the mapping torus of f^* and we denote the suspension flow by φ^*. Our assumptions on the singularities of f imply that $\varphi^*/\partial M^*$ is a Morse–Smale flow such that each boundary component C has 2 or 4 parallel periodic orbits.

We will construct a new flow φ by blowing down φ^*. For each C we can find a cross section K_C that is a circle and meets the periodic set of φ^*/C four times. The return map $r_C: K_C \circlearrowright$ is Morse–Smale with 4 periodic points. There is a closed 1-form ω_C on C that vanishes on K_C and is positive on the vector field $d\varphi^*/dt$. We blow down M^* by collapsing each leaf of the circle foliation defined by ω_C to a point, for all C, and obtain a closed 3-manifold M. By reparametrizing φ^* so that $\omega_C(d\varphi^*/dt) = 1$ everywhere on C, we obtain an induced flow φ on M. Note that M and φ depend on the cohomology classes of the forms ω_C but that the topological type of φ is determined by these classes alone[3].

The relation between f and φ is as follows. By choosing ω_C to be nonsingular on $S^* \cap C$ for all C, the image Σ of S^* in M satisfies the following property: Σ is an immersed surface whose interior is embedded and transverse to φ and $\partial\Sigma$ consists of closed orbits and there is a $t > 0$ such that every trajectory meets Σ in any time interval of length t. We say that such a Σ is a *surface of section* for φ: our usage differs from that of [1] only in that we allow the boundary to be immersed rather than embedded. There is a first return map for the interior points of Σ that agrees with f^*. So f is obtained from φ by blowing down the return map for a surface of section.

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In our situation the dynamics of φ near $\partial\Sigma$ is clearly that of a hyperbolic periodic orbit. We may choose a smooth structure on M so that φ is given by a smooth vectorfield and these orbits are hyperbolic (e.g. by isotoping the return map for a smooth Markov partition [6] near $\partial\Sigma$ while preserving hyperbolicity). It is clear that φ is a transitive Anosov flow. We will soon prove

THEOREM 1. *Any transitive Anosov flow on a closed* 3*-manifold is topologically conjugate to one constructed from a pseudo-Anosov map as above.*

We remark that the flow φ is constructed by Dehn surgery on a non-Anosov flow, namely the suspension flow of the pseudo-Anosov map f. In this sense our construction is an extension of that in [4]. It has the technical advantage that one never loses track of the stable and unstable foliations.

§2. SURFACES OF SECTION FOR ANOSOV FLOWS

It is clear that Theorem 1 has the following corollary.

THEOREM 2. *Every transitive Anosov flow φ on a closed* 3*-manifold M has a surface of section.*

We now show that the first theorem reduces to the second, which we will then prove.

THEOREM 3. *Let φ be a transitive Anosov flow on a closed* 3*-manifold M. If φ admits a surface of section then φ is topologically conjugate to a flow of the sort constructed in* §1.

Proof. We will analyze the return map $r : \Sigma\circlearrowleft$ associated to the surface of section Σ. First we blow up φ along $\partial\Sigma$, obtaining a manifold M^* (the exterior of the link $\partial\Sigma \subset M$). After a perturbation we may suppose that $\Sigma \subset M^*$ is a global cross-section for the blown up flow φ^* [3]. The return map r preserves a pair of transverse foliations arising from the intersections of Σ with the stable and unstable foliations for φ. One sees that these foliations have no singularities on the interior of Σ but have thorn-type singularities on the boundary. Using the symbolic dynamics for φ, we can construct transverse measures for these foliations that scale by a nontrivial proportionality constant when r is applied (see [2], Exposé 14). Thus r is topologically conjugate to a pseudo-Anosov map. It follows immediately that φ is topologically conjugate to an example arising from the construction of §1. Q.E.D.

Proof of Theorem 2. Given a periodic orbit γ for φ, Ratner constructed a Markov family $\mathcal{M}$ of smooth rectangles transverse to φ whose boundaries lie in the stable and unstable manifolds of γ [6]. Choose a point p in the interior of one of these rectangles $R \in \mathcal{M}$. We fix an orientation on the local stable and unstable manifolds of p and regard unstable leaves as horizontal and stable leaves as vertical.

As φ is transitive we may find q near p, but above and to the right of p, such that $q \notin \gamma$, q is periodic and the local Poincare map for q preserves local stable and unstable orientations. We likewise choose r to the lower left of p with the same properties.

The periodic symbol sequences for q and r both contain an R: by combining these into a single periodic sequence we produce a periodic point $s \in R$ for φ that alternately follows r for one period and follows q for one period. In terms of canonical coordinates, s is very near $[r, q]$ (assuming, as we may, that q and r have large periods). Corresponding to the second occurence of R in the symbol sequence for s is a point s' in the orbit of s with s' very near $[q, r]$.

Now we join q to s' by a straight path γ_1 and we let γ_2 be the image of γ_1 under the Poincare map for q. Then γ_2 is a path in R joining q to s. We likewise join r to s by a straight path ϵ_1 and let ϵ_2 (joining r to s') be the image of ϵ_1 under the Poincare map for r. With the choices made above, ϵ_1, γ_2, γ_1 and ϵ_2 bound a diamond shaped region D in R with $p \epsilon$ int D. We let E and G be the immersed strips tangent to the flow that connect ϵ_1 to ϵ_2 and γ_1 to γ_2, as in Fig. 1.

We now round off $D \cup E \cup G$ to an immersed disc with 2 holes Σ_p whose boundary consists of the closed orbits through q, r, and s and whose interior contains p and is transverse to φ. Near D, Σ_p is shaped like a saddle with 4 corners, much like the Scherk minimal surface $z = \log(\sin y/\sin x)$.

By using various Markov partitions we can construct such a Σ_p for any $p \in M$. As M is compact we can find a finite set P such that every segment $\varphi_t m$, $0 \leq t \leq 1$, $m \in M$, meets int Σ_p for some $p \in P$. We may assume these Σ_p's chosen so that $\partial \Sigma_p$ are disjoint for all $p \in P$. The immersed surface $\Sigma' = \bigcup_{p \in P} \Sigma_p$ has embedded boundary, its interior is transverse to φ and it has the return property. We must construct an embedded surface with these properties.

We first perturb Σ' on its interior, keeping it transverse to φ, so that its self-intersections are in general position: denote this new surface Σ''. The self-intersections of Σ'' consist of finitely many curves of double points, possibly ending at the boundary, and finitely many interior triple points. Using the normal orientation furnished by φ, we cut and paste Σ'' to obtain an embedded surface Σ with the required properties (see Fig. 2). Q.E.D.

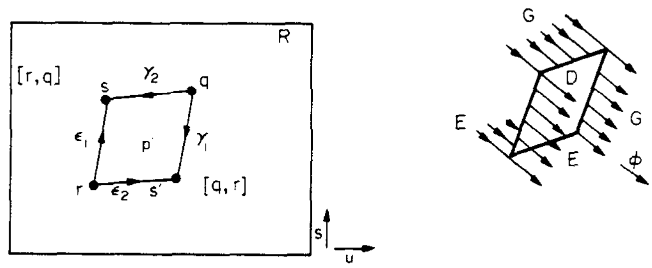

Fig. 1.

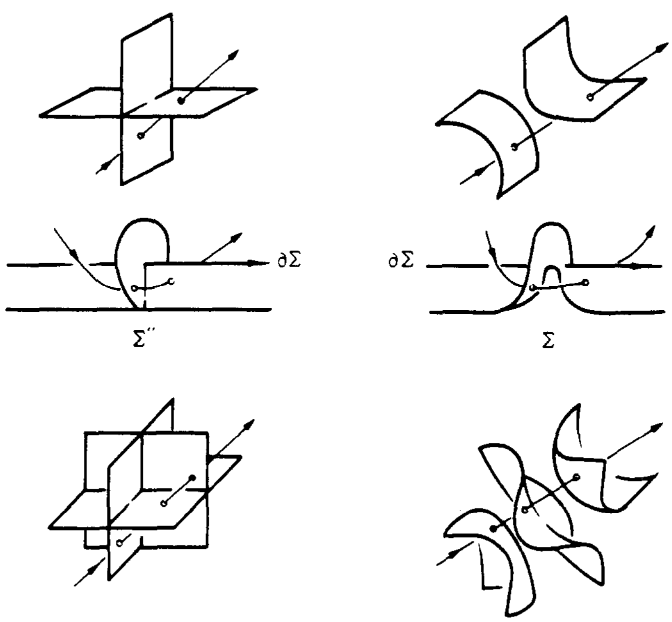

Fig. 2.

Note that we found an embedded surface of section: the added generality of our definition in §1 was helpful for constructing flows but isn't needed to construct sections.

Some understanding of the preceding proof can be obtained by considering the homological criteria for surfaces of section in [3]. There we showed that a finite set of orbits bounds a surface of section if all other orbits wind suitably about the given ones. One can check directly that $\partial\Sigma'$ meets this criterion from the known properties of Σ' and so deduce the existence of Σ without the explicit construction given above.

§3. AN EXAMPLE OF BIRKHOFF'S AND SOME QUESTIONS

We will apply some methods of [1] to the geodesic flow G on a closed surface X of negative curvature.

Choose $g_1, g_2, \ldots, g_k$ to be simple closed geodesics that divide X into simply connected pieces and such that for some $t > 0$ every geodesic meets $\cup g_i$ in any time interval of length t. Then Birkhoff gave a beautiful geometric construction of a surface of section for G with $2k$ boundary components, corresponding to the closed orbits of G obtained by lifting the g_i's (with each of the two possible orientations) to the unit tangent bundle. That a system of curves g_i with the desired properties exists is easily seen (see Fig. 3, also from [1]).

For the example illustrated in the figure, one finds the surface of section as follows. Let R be the disc with 6 corners on the bottom front on the surface. Form a nested family of curves that fills R with a singularity in the interior. Do the same for the region R' on the top back of the surface. Consider all unit vectors tangent to these systems of curves and let $\Sigma \subset T_1X$ be their closure. Then Σ is a surface with the desired boundary. If the curves are chosen to be strictly convex then the interior of Σ is transverse to G and Σ is a surface of section for G (it is a pleasant exercise to verify that every sufficiently long geodesic flowline meets Σ).

In this example, Σ is constructed by gluing together 4 punctured discs corresponding to the tangent vectors to the smooth curves. The intersection points of the g_i's each give a pair of 1-handles. By gluing these on one obtains two genus 0 surfaces each with 6 holes corresponding to the g_i's and 2 punctures corresponding to the singularities in the center of R and R'. As shown in Fig. 4, adjoining the circles of unit tangent vectors at these singularities glues these sheets together to give a genus 1 surface with 12 holes as Σ.

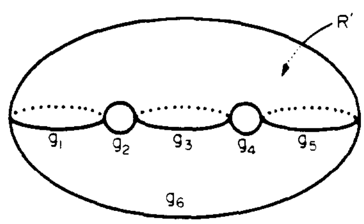

Fig. 3.

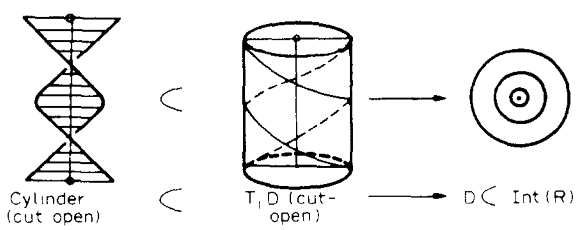

Fig. 4.

The return map for this Σ was known by Birkhoff to have infinitely many periodic points (corresponding to the closed geodesics) and to be area preserving. As Birkhoff delivered his paper in 1916, this is probably the first pseudo-Anosov map that arose in dynamical systems, preceding Teichmuller by about 25 years and Thurston by over 50!

In conclusion we point out that the section of Theorem 2 is far from unique when it exists. However if L is its boundary link then it is determined up to isotopy by its cohomology class in $H^1(M - L; \mathbb{Z})$[3]. We do not know at present whether all the Anosov flows constructed in §1 are obtained from the suspension of a toral Anosov diffeomorphism by surgery. The Birkhoff section constructed above shows that the geodesic flow arises in this way, but our methods of §2 do not control the genus of the section.

Perhaps the existence of sections for the flows studied above can be used to show that they are topologically conjugate to flows that preserve a smooth volume or that have smooth Anosov foliations.

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